FACTORING QUADRATIC EQUATIONS

Summary

The goal of this section is to summarize the methods allowing us to factor quadratic equations, i.e. of form $ax^2 + bx + c$. We will avoid using the famous discriminant formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

as much as possible. Three methods allow us to carry out the factoring of most quadratic functions.

1. Difference of squares

There is a formula that allows for rapid factorization. When a function presents in the form $x^2 - k^2$, it can be factored by the *difference of squares* formula, i.e.

$$x^2 - k^2 = (x - k)(x + k).$$

Hence, you need to obtain the square root of the first and second term and multiply their sum with their difference. The negative sign separating the terms $x^2$and k^2 is of capital importance. For example, the rule presented above cannot be applied to $x^2 + k^2$, as the name *difference* in squares implies!

Example

$x^2 - 9 = x^2 - (3)^2 = (x - 3)(x + 3)$

$x^2 - 100 = x^2 - (10)^2 = (x - 10)(x + 10)$

$x^2 - 81 = x^2 - (9)^2 = (x - 9)(x + 9)$

In the preceding examples, we chose to use values that are perfect squares. This criteria is not generally necessary. For example, although 8 is not a perfect square, its root is well defined (it exists) and is equal to $\sqrt{8}$.

Example

$x^2 - 8 = x^2 - (\sqrt{8})^2 = (x - \sqrt{8})(x + \sqrt{8})$

$x^2 - 3 = x^2 - (\sqrt{3})^2 = (x - \sqrt{3})(x + \sqrt{3})$

$x^2 - 45 = x^2 - (\sqrt{45})^2 = (x - \sqrt{45})(x + \sqrt{45})$

Also, it is possible that the first term is not x^2 but $9x^2$. Once again, you do not need to worry. We must find the square root of $9x^2$ instead of the square root of x^2. In all other aspects, the method remains the same.

Example

$9x^2 - 25 = (3x)^2 - (5)^2 = (3x - 5)(3x + 5)$

$16x^2 - 49 = (4x)^2 - (7)^2 = (4x - 7)(4x + 7)$

$8x^2 - 9 = (\sqrt{8}x)^2 - (3)^2 = (\sqrt{8}x - 3)(\sqrt{8}x + 3)$

2. Simple factorization

We must avoid confusing the form of the difference of squares x^2 - k^2 with ax^2 - bx. The presence of an x in the second term will allow us to proceed with a simple factorization. We can bring out the *common factor*, i.e. the *x*.

Example

$x^2 - 6x = x(x - 6)$

We *factored* (placed in front of a parenthesis) an x since each of the terms $x^2 - 6x$ has at least an x. Note that the product $x(x - 6)$ is equal to $x^2 - 6x$. You can convince yourself by distributing (multiplying) the x to each of the terms of the expression$(x - 6)$. You can interpret the factorization as the operation of separating all the common factors to the terms of the expression.

Example

Factor $25x^2 - 10x$

What are all the factors common to $25x^2 - 10x$? Each has at least one x. The coefficients 25 and 10 also have the common factor 5. $5x$ constitutes the greatest common factor of $25x^2$ and $10x$. If $5x$ is removed from these terms (or better factorized), what remains of $25x^2$ and $10x$? Of $25x^2$, $5x$ will remain. Of $10x$, 2 will remain. Consequently,

$25x^2 - 10x = 5x(5x - 2)$

Example

Factor the following quadratic functions.

$x^2 + 12x = x(x + 12)$

$2x^2 + 12x = 2x(x + 6)$

$-36x^2 + 6x = 6x(-6x + 1)$ or-$6x(6x - 1)$

$225x^2 - 45x = 15x(15x - 3)$

$9x^4 - 16x^2 = x^2(9x^2 - 16) = x^2(3x - 4)(3x + 4)$

Note that in example 5, we used a factorization followed by a difference of squares… nothing stops us from using or combining two factoring methods for one problem. You must therefore be at ease with all the methods we suggest here.

3. Compounded factorization

Consider the quadratic equation $x^2 + 5x + 6$. It is clearly not a problem that can be solved by the difference in squares, its form not being suited for this method. Also, you will notice that no common factor can be found. Let us slightly adjust the quadratic function

$$x^2 + 5x + 6 = x^2 + 2x + 3x + 6$$

Note that we did not cheat : even though we rewrote the function, its total value did not change. With this new presentation, notice that a factoring can be carried out to the pairs of terms $x^2 + 2x$ and $3x + 6$. x is the common factor of $x^2 + 2x,$ and 3 is common factor of $3x + 6$. By proceeding to a simple factoring, pair by pair, we obtain

$x^2 + 2x + 3x + 6 = x(x + 2) + 3(x + 2)$.

But, this is not all! Now, in this new expression, $(x + 2)$ is a common factor. We can therefore factor out $(x + 2)$. Thus, from $x(x + 2)$, x remains. From $3(x + 2)$, 3 remains. Consequently,

$$x(x + 2) + 3(x + 2) = (x + 2)(x + 3)$$

Note that we carried out three consecutive factorizations in order to complete the factoring, as the name compounded factorization implies.

One aspect of the method remains unexplained: why rewrite $\boldsymbol{x^2 + 5x + 6}$ under the form $x^2 + 2x + 3x + 6$ instead of $x^2 + x + 4x + 6$ or $x^2 + 9x - 4x + 6$? Here is the process allowing us to discover which way to separate the central term (multiple of *x*) :

Given $ax^2 + bx + c$, a quadratic function. The term bx must be separated to obtain the complete factorization. To do this, we must find two numbers, m and n, such that :

1. the product $mn = a$c

2. the sum $m + n = b$

Once the numbers m and n are found, replace the expression $ax^2 + bx + c$ by $ax^2 + mx + nx + c$ and proceed by compounded factorization.

Example

Factor the quadratic function $x^2 + 5x + 6$.

The coefficients are respectively

$a = 1$

$b = 5$

$c = 6$

Since $b \neq 0$ and $c \neq 0$, we must first separate the central term 5*x* into two parts. To do this, we must find two numbers, m and n, such that

1. the product $mn = ac = (1)(6) = 6$

2. the sum $m + n = b = 5$

Which two numbers have product 6 and sum 5? By trial and error, you will find that $m = 2$ and $n = 3$. We use these two numbers to substitute $5x$ by $2x + 3x$. Thus, $x^2 + 5x + 6 = x^2 + 2x + 3x + 6$. Then we use compounded:

$$x^2 + 5x + 6 = x^2 + 2x + 3x + 6 = x(x + 2) + 3(x + 2)$$
$$= (x + 2)(x + 3)$$

Example

Factor the quadratic function $6x^2 - 17x + 12$.

The coefficients are respectively

$a = 6$

$b = 17$

$c = 12$

Since $b \neq 0$ and $c \neq 0$, we must first separate the central term, $-17x$ into two parts. To do this, we must find two numbers, m and n, such that

1.the product $mn = ac = (6)(12) = 72$

2. the sum $m + n = b = -17$

What are the two numbers whose product is 72 and the sum is -17 ? You will find that $m = -8$ and $n = -9$. Note that the negative signs are very important since we want a sum of -17. We use these two numbers to substitute $-17x$ by $-8x - 9x$. Thus,

$$6x^2 - 17x + 12 = 6x^2 - 8x - 9x + 12$$

We can now pursue compounded factorization, by beginning by finding the common factors to each pair of terms: 2*x* is a common factor to $6x^2 - 8x$; -3 is a common factor to $-9x + 12$ (we must generally make sure to factor the negative sign of *x* if it has one). Then we complete the factoring.

$$6x^2 - 17x + 12 = 6x^2 - 8x - 9x + 12 = 2x(3x - 4) - 3(3x - 4)$$
$$= (3x - 4)(2x - 3)$$

Example

Factor the quadratic function $3x^2 - x - 10$.

The coefficients are respectively

$a = 3$

$b = -1$

$c = -10$

Since $b \neq 0$ and $c \neq 0$, we must first separate the central term, $-x$ into two parts. To do this, we need to find two numbers, m and n, such that

1. the product $mn = ac = (3)(-10) = -30$

2. the sum $m + n = b = -1$

Which two numbers have product -30 and sum -1? The only possible combination is $m = -6$ and $n = 5$. Note that the signs must be chosen to correspond to the criteria of product and sum … We use these two numbers to substitute -*x* by -6*x* + 5*x*. Therefore,

$$3x^2 - 1x + 12 = 3x^2 - 6x + 5x - 10$$

We can now proceed by compounded factorization, i.e. by first finding the common factors to each pair of terms :

$3x$ is a common factor of $3x^2 - 6x$;

5 is a common factor of $5x - 10$.

We complete the factoring

$$3x^2 - x - 10 = 3x^2 - 6x + 5x - 10 = 3x(x - 2) + 5(x - 2)$$
$$= (x - 2)(3x + 5)$$

4. Exercises

Factor the following quadratic functions:

1. $4x^2 - 25$
2. $3x^2 - 12x$
3. $2x^2 - 8$
4. $3x^2 + 7x - 6$
5. $x^2 - 3x - 28$
6. $x^2 - 12x + 36$
7. $8x^2 + 7x - 1$
8. $4x^2 - 4x + 1$
9. $10x^2 - 3x - 4$
10. $x^2 - 9x + 20$

Solutions

1. $(2x - 5)(2x + 5)$
2. $3x^2 - 12x = 3x(x - 4)$
3. $2(x^2 - 4) = 2(x - 2)(x + 2)$
4. $(3x - 2)(x + 3)$
5. $(x - 7)(x + 4)$
6. $(x - 6)(x - 6) = (x - 6)^2$
7. $(8x - 1)(x + 1)$
8. $(2x - 1)(2x - 1) = (2x - 1)^2$
9. $(2x + 1)(5x - 4)$
10. $(x - 5)(x - 4)$