MATHEMATICAL MODELLING

Summary

1. The concept of "variables"

An algebraic expression (usually one single letter: $, y, z, \ldots$) is called a "variable" if it replaces an unknown value (physical, economical, temporal, etc.).

The variable's role is to occupy the position that would take a value, if it were available.

Example:

If the tax rate is 40 %, what amount of taxes will a person need to pay?

Solution:

Answering such a question requires knowledge of the annual salary of the person. Since the salary is unknown, we replace it with a variable. For example, if we define the variable

$x =$ the person's annual salary.

He will have to pay 40 % of x in taxes, i.e.

$$Taxes = 40\,\%\,x = 0{,}40\,x$$

The variable x therefore occupies position of the salary in the calculation for taxes. That is, until the value of the salary is known.

2. Mathematical modeling

The process by which we use mathematical expressions to describe a real quantitative situation is called modeling. Modeling consists of writing in mathematical terms what is first expressed in words, using variables where necessary. The preceding example illustrates modeling: even though we do not know the salary of the individual, we were nonetheless able to obtain an expression that adequately represents the taxes paid by him.

Example 1:

Marc wishes to invest in a stock with a 10% return annually. How much money will he have at the end of the year?

Solution:

Marc's initial investment is unknown. Let us define:

x : the amount that Marc invests in this share

The amount accumulated at the end of the year will be

$$x + (10\,\%)\,x = x + 0{,}1\,x = 1{,}1\,x.$$

Example 2:

A carpenter produces and sells his own furniture. Pine tables are sold for 650 \$, cherry tables for 750 \$ and maple tables for 850 \$. What is the carpenter's annual revenue?

Solution:

The annual revenue of the carpenter can only be obtained if the amount of tables sold of each type is known. Variables must therefore replace these quantities, all unknown for the moment. Let us define:

x: the number of pine tables sold during the year
y: the number of cherry tables sold during the year
z: the number of maple tables sold during the year

Each pine table produces a revenue of 650 \$. If x pine tables are sold, a revenue of 650 times x will be obtained. The same argument applies to the other types of tables. Consequently,

$$Total\ revenue = 650\,x + 750\,y + 850\,z$$

Example 3:

The three phases of a project must be undertaken sequentially, which means that one phase cannot begin before the previous phase is finished. We know that the cost of each of the phases breaks down into a fixed cost, independent of its duration, and a variable cost, which depends on the duration. The following table summarizes the situation:

PHASE	**1**	**2**	**3**
FIXED COST	318 000 $	212 000 $	220 000 $
VARIABLE COST	15 000 $ / day	14 000 $ / day	16 000 $ / day

The designer of the project must propose a price for the project. He would like to set a price that ensures a profit margin of at least 10%. Express the total cost of the project and the price the designer should propose in function to the duration of each phase.

Solution:

The duration of each phase is unknown. Let us thus define the three following variables:

$x = duration\ of\ phase\ \ 1\ (in\ days)$
$y = duration\ of\ phase\ 2\ (in\ days)$
$z = duration\ of\ phase\ 3\ (in\ days)$

The cost of phase 1 can be broken down in a fixed cost (318 000 $) and a variable cost (15 000 $ per day).
If phase 1 lasts x days, the cost of this phase will be

$$Cost_{phase1} = 318\,000 + 15\,000\,x.$$

The same principle applies to the two other phases.
The total cost of the project can be expressed as the sum of the costs of the three phases:

$$C_T = Total\ cost\ of\ the\ project = Cost_{phase1} + Cost_{phase2} + Cost_{phase3}$$
$$= (318\,000 + 15\,000\,x) + (212\,000 + 14\,000\,y) + (220\,000 + 16\,000\,z$$
$$= 15\,000\,x + 14\,000\,y + 17\,600\,z + 750\,000$$

The proposed price for the project has to ensure a 10% profit margin for the designer.

The price must therefore be at least 10% greater than the total cost:

$$Prix \geq C_T + 10\%C_T = 1.1C_T = 1.1(15\,000x + 14\,000y + 16\,000z + 750\,000)$$
$$Prix \geq 16\,500x + 15\,400y + 16\,000z + 825\,000$$

Example 4:

A square bottom box without cover is made from a material that costs \$ 0.75 per square metre for the sides and \$ 0.95 per square metre for the bottom. Express the total cost of the material required to construct the box in function of its width and height.

Solution:

To calculate the cost of material required, we must establish the surface area of each side of the box and of its bottom (in square metres). The dimensions of the box are unknown at this time.

Let us define:

$x = length\ of\ the\ side\ of\ the\ bottom\ of\ the\ box\ (in\ metres)$

$h = height\ of\ the\ box\ (in\ metres)$

The four sides of the box have an area of xh square metres each. Therefore, each side costs $0.75\ xh$ in material.

The bottom of the box has an area of $x \times x = x^2$ square metres. The cost of material for the bottom is $0.95x^2$. Therefore, the total cost for the material needed to construct a box is given by:

$$Total\ cost = 0.95\, x^2 + 4(0.75\, xh) = 0.95\, x^2 + 3\, xh$$

There are problems that have constraints that intervene between different variables. Let us take the context of example 2. Here, the values of x, y and z are perfectly free. However, a shortage of maple wood could oblige the carpenter to produce half as many maple tables as cherry tables. This constraint can also be modeled in mathematical notations using the expression $z = (1/2)\ y$ or even, $y = 2\ z$.

When a problem is modeled, it is important to consider all given information. This way, all magnitudes (physical, economical, temporal) and all constraints must be translated into mathematical language.

Example 5 :

A farmer is looking to divide to plant different cultures. Traditionally, corn fields returned \$ 3.50 \$ per square metre. Oat fields returned \$ 2.75 per square metre. Orchards produced revenues of \$ 4.50 per square metre. The farmer has a land of 1 million square metres. In order to feed his farm animals, the cultivator must dedicate a minimum of 300 000 square metres to the culture of corn and oats (together). However, since corn is more susceptible to long periods of drought, he does not want this culture to occupy more than 200 000 square metres. Lastly, he would like to allot the same amount of space to oats and orchards.

Which expression correctly represents the revenues of the farmer? Model all constraints that the farmer must respect.

Solution

Three unknowns must be identified in order to model this problem completely:

$x :\ the\ surface\ alotted\ to\ corn\ (m^2)$

$y :\ the\ surface\ alotted\ to\ oats\ \ (m^2)$

$z :\ the\ surface\ alotted\ to\ apples\ (m^2)$

The revenues are expressed in function to the surface occupied by each of the cultures and the revenues these return per square metre

$$Revenues\ =\ 3{,}50\ x\ +\ 2{,}75\ y\ +\ 4{,}50\ z$$

Four constraints are forced on the farmer:

1. "The farmer has a land of 1 million square metres"

$$x + y + z \leq 1\,000\,000$$

2. "The farmer must dedicate a minimum of 300 000 square metres to corn and oats"

$$x + y \geq 300\,000$$

3. "Since corn is more susceptible to long periods of drought, he does not want this culture to occupy more than 200 000 square metres"

$$x \leq 200\,000$$

4. "He would like to allot the same space to oats and orchards "

$$y = z$$