Matrix generalities

Summary

Matrix of dimension $m \times n$; $\mathrm{A} = (\mathrm{a_{ij}}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$

1. Particular matrices

Zero matrix:

All its elements $\mathrm{a_{ij}} = 0$

Square matrix of order n:

Number of lines = number of columns = n

Diagonal matrix:

$$\begin{pmatrix} \mathrm{a}_{11} & 0 & \cdots & 0 \\ 0 & \mathrm{a}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{a}_{mn} \end{pmatrix}$$

Identity matrix of order:

$$\mathrm{I}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Upper triangular matrix:

$$\begin{pmatrix} \mathrm{a}_{11} & \mathrm{a}_{12} & \cdots & \mathrm{a}_{1n} \\ 0 & \mathrm{a}_{22} & \cdots & \mathrm{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{a}_{mn} \end{pmatrix}$$

Lower triangular matrix:

$$\begin{pmatrix} \mathrm{a}_{11} & 0 & \cdots & 0 \\ \mathrm{a}_{21} & \mathrm{a}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{a}_{m1} & \mathrm{a}_{m2} & \cdots & \mathrm{a}_{mn} \end{pmatrix}$$

2. Matrix operations

Scalar multiplication:

$$kA = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}$$

Sum of two matrices of the <u>same dimension</u> *$(m \times n)$ $A = (a_{ij})$ and $B = (b_{ij})$*

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{31} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$$

Multiplication of two matrices A and B of dimensions $m \times n$ and $\times p$:

$$AB = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m1} & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix} =$$

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mj} & & c_{mp} \end{pmatrix} = C\ (dimension\ m \times p)$$

with

$$\mathrm{c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{n1}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}\ (i = 1,2,\ldots,m; j = 1,2,\ldots,p)}$$

WARNING: The product AB is only defined if the number of columns of matrix « A » is equal to the number of lines of matrix « B ». Moreover, in general $AB \neq BA$.

Transposition *(A^T or $A^{'}$) :*

The transpose of matrix A is obtained by replacing the lines of a matrix by its columns. If the matrix A has dimensions $m \times n$, the transpose AT, will have dimensions $n \times m$.

$$\mathrm{A^T} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

Properties: Given A and B are two matrices and k a scalar

1. $\mathrm{(A + B)^T = A^T + B^T}$
2. $\mathrm{(A^T)^T = A}$
3. $\mathrm{(kA)^T = kA^T}$
4. $\mathrm{(A\,B)^T = B^T A^T}$

For all matrices A, the product $A^T A$ is a symmetrical square matrix and the elements of its principal diagonal are not negative.

Trace of a square matrix of order n, $A = (a_{ij})$ (written $tr(A)$):

Sum of the elements of the principal diagonal, i.e. $\mathrm{tr(A)} = \mathrm{a_{11}} + \mathrm{a_{22}} + \cdots + \mathrm{a_{nn}}$

Properties:
1. $\mathrm{tr(A + B) = tr(A) + tr(B)}$
2. $\mathrm{tr\,(cA) = c\,tr(A)}$

3. Row echelon form of a matrix

A matrix $\mathrm{A} = (\mathrm{a_{ij}})$ is called « **row echelon** » if the number of « 0 » preceding the first non-zero element of a line increases line by line.
It is called « **reduced row echelon form** » if, in addition, the first non-zero element of a line is equal to « 1 » and if, in the corresponding column (pivot column), all other elements are « 0 ».

We can reduce a matrix to its row echelon form (or reduced row echelon form) by carrying out elementary operations on its lines:
- Multiply one line by a non-zero scalar.
- Intervene or permutate 2 lines.
- Add « k » times another line to a line.

4. Rank of a matrix $r(A)$

The rank of a matrix A with dimensions $m \times n$ corresponds to the number of non-zero lines of its reduced row echelon form. It is said that A is of « full rank » if $\mathrm{r(A) = m}$

Note: The rank of a matrix gives the maximum number of linearly independent lines as well as the maximum number of its linearly independent columns.

Properties:
1. If B can be obtained from A by successive applications of elementary operations on its lines, then $\mathrm{r(A) = r(B)}$
2. $\mathrm{r(A^T) = r(A)}$
3. If the matrix product AB is defined, then $\mathrm{r(AB)} \leq \min\{\mathrm{r(A); r(B)}\}$

5. Inverse matrix

Given A a square matrix $n \times n$. The inverse of A (written A^{-1}) , if it exists, is the matrix that satisfies

$$\mathrm{AA}^{-1} = \mathrm{A}^{-1}\mathrm{A} = \mathrm{I_n}$$

If the inverse of A exists, we can obtain it in the following manner:

1. Consider the augmented matrix

2. $(\mathrm{A} \vdots \mathrm{I}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots & 0 & 1 & \cdots & 0 \\ & & \ddots & & \vdots & & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} & \vdots & 0 & 0 & \cdots & 1 \end{pmatrix}$

3. Carry out the elementary operations on the lines of the augmented matrix until it becomes $(\mathrm{I} \vdots \mathrm{B})$. The matrix B is then the inverse of A i.e. $\mathrm{B} = \mathrm{A}^{-1}$.

Properties:

1. If A is invertible, then $A - 1$ is also invertible and $(\mathrm{A}^{-1})^{-1} = \mathrm{A}$.
2. If A is invertible, then $(\mathrm{A}^{-1})^{\mathrm{T}} = (\mathrm{A}^{\mathrm{T}})^{-1}$
3. If A and B are 2 invertible square matrices with the same dimensions, then their product AB is also invertible and $(\mathrm{A\,B})^{-1} = \mathrm{B}^{-1}\mathrm{A}^{-1}$

Existence: A of dimension $n \times n$ is invertible if $\mathrm{r(A)} = \mathrm{n}$

6. Determinant ($det(A)$ or $|A|$)

Given A a square matrix $\mathrm{n} \times \mathrm{n}$.

Matrix 2 × 2 : $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$

Higher order: The determinant is equal to the sum of the products obtained by multiplying the elements of whichever line (or a column) by their respective cofactors = $\mathrm{A_{ij}} = (-1)^{\mathrm{i+j}}|\mathrm{M_{ij}}|$ where M_{ij} (the minor) is the square sub-matrix $(\mathrm{n}-1) \times (\mathrm{n}-1)$ obtained by suppressing the i$^{\text{th}}$ line and the j$^{\text{th}}$ column of A.

Thus $|\mathrm{A}| = a_{11}\mathrm{A_{i1}} + a_{22}\mathrm{A_{i2}} + \cdots + a_{nn}\mathrm{A_{in}}$.

Properties :

1. If A has a line (or column) of « 0 », then $|\mathrm{A}| = 0$.

2. If A has 2 identical lines (or columns), then $|\mathrm{A}| = 0$.

3. If A is triangular, then $|A| =$ product of its diagonal elements. In particular, $|\mathrm{I_n}| = 1$.

4. If B is obtained from A by multiplying one single line (column) by a scalar k, then $|\mathrm{B}| = \mathrm{k}|\mathrm{A}|$.

5. If B is obtained by permutation of 2 lines (or columns) of A, then $|\mathrm{B}| = -|\mathrm{A}|$.

6. If B is obtained from A by adding the multiple of a line (column) to another, then $|\mathrm{B}| = |\mathrm{A}|$.

7. $|\mathrm{A^T}| = |\mathrm{A}|$

8. If A and B are 2 square matrices of the same dimension, then $|\mathrm{AB}| = |\mathrm{A}||\mathrm{B}|$.

9. A is invertible if $|\mathrm{A}| \neq 0$. We then say that the matrix is non-singular.

7. Conjugate transpose

Given A a square matrix of order n. The conjugate transpose of A (written adj(A)) is defined as the transpose of the matrix of cofactors of A i.e.

$$\mathrm{A} = \begin{pmatrix} \mathrm{A_{11}} & \mathrm{A_{12}} & \cdots & \mathrm{A_{1n}} \\ \mathrm{A} & \mathrm{A_{22}} & \cdots & \mathrm{A_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{A_{m1}} & \mathrm{A_{m2}} & \cdots & \mathrm{A_{mn}} \end{pmatrix}$$ where $\mathrm{A_{ij}} = (-1)^{\mathrm{i+j}}|\mathrm{M_{ij}}|$ (cofactor, see previous page)

If A is a square matrix such that $|\mathrm{A}| \neq 0$, then A is invertible and $\mathrm{A^{-1}} = \frac{1}{|\mathrm{A}|}\mathrm{adj(A)}$.

8. Positive-definite matrix

A matrix $n \times n$ symmetric A is said to be « positive definite » if the product $\mathrm{X^TAX} > 0$ for all vectors X $(\mathrm{n} \times 1)$.
It is « positive semi-definite » if $\mathrm{X^TAX} \geq 0$ for all X.

A matrix $n \times n$ symmetric A is said to be « negative-definite » if the product $\mathrm{X^TAX} < 0$ for all vectors X $(\mathrm{n} \times 1)$.
It is « negative semi-definite » if $\mathrm{X^TAX} \leq 0$ for all X.

9. Linear equation systems as matrix equations

All systems of linear equations (m equations, n unknowns) :

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{gathered}$$

Can be written as the matrix equation:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$ or simply as $AX = B$