QUADRATIC, EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Content

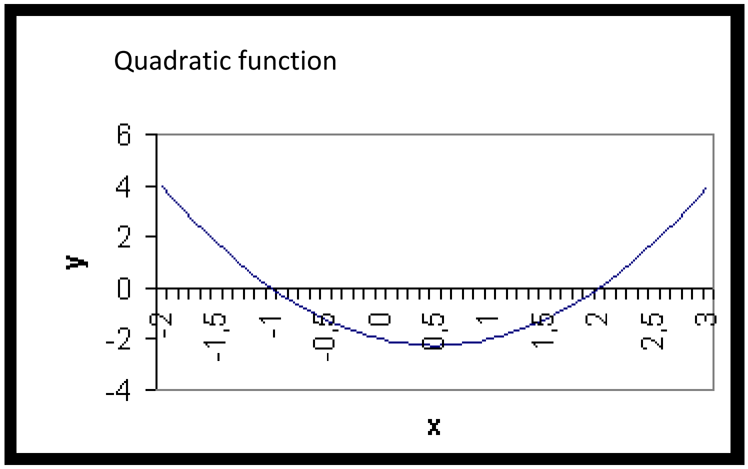
1. Parabolas

We call polynomials of the second degree parabolas or quadratic functions. One can recognize a parabola because of the form of its equation :

$$y = ax^2 + bx + c$$

Without wanting to go into too much detail, it is important to be able to sketch a quadratic function with sufficient precision.

In the past, you have certainly seen the particular form of a parabola, characterized by its top and its "wings"…



How can we obtain the characteristics of a parabola so that we can sketch it?

The graph of a parabola can easily be sketched if we know the following information:

- Where is the top of the parabola?
- Is the parabola open towards the top or the bottom?
- What is the intercept of the parabola?
- Does the parabola have roots (zeros)?

1.1. Top of a parabola

Given $y = ax^2 + bx + c$, the equation of a some parabola.

The value of x at the top is $x = \frac{-b}{2a}$.

The corresponding y-value can be obtained by substitution of x in the equation of the parabola.

1.2. Orientation of a parabola

The orientation of a parabola is determined by the sign of "a", the coefficient of x^2.

If $a > 0$ then the parabola is open towards the top.

If $a < 0$ then the parabola is open towards the bottom.

Note that if the parabola is open towards the top then its top is a minimum, but if it is open towards the bottom, its top is a maximum.

1.3. Intercept of a parabola

We have learnt in the section on linear equations that the intercept is the value of a function when $x = 0$.

In the case of a parabola with equation $y = ax^2 + bx + c$, if $x = 0$ then $y = c$.

The point $(0, c)$ is hence the intercept of the parabola.

1.4. Roots (or zeros) of a parabola

<u>**Definition**</u> : Each value of x such that the function $y = f(x)$ takes the value 0 is called a **root**.

To find the roots of a parabola $y = ax^2 + bx + c$, we need to solve the equation $ax^2 + bx + c = 0$. One can proceed by factorization or use the formula :

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example

Trace the graph of the parabola with equation $y = 2x^2 + 3x - 5$.

Top of the parabola:

The top is situated at the position $x = \frac{-b}{2a} = \frac{-3}{2(2)} = \frac{-3}{4} = -0{,}75$.

And the corresponding y-value is $y = 2(-0{,}75)^2 + 3(-0{,}75) - 5 = -6{,}125$

The top is hence at the point$(-0{,}75\,;\,-6{,}125)$

The parabola is open towards the top because $a > 0$.

The intercept is the value of y when $x = 0$. In this case,

$y = 2(0)^2 + 3(0) - 5 = -5$.

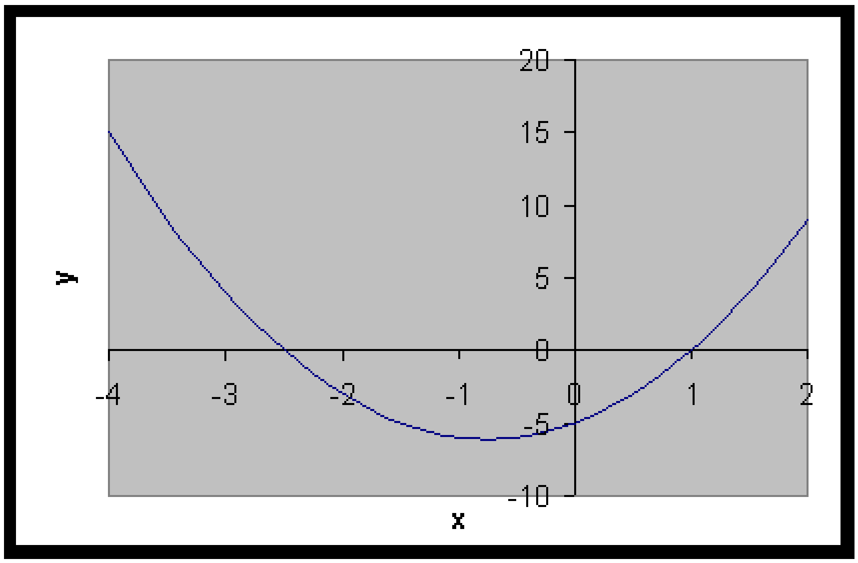
The point $(0, -5)$ is thus the intercept.

Using the *discriminant* formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we obtain:

$$\frac{-3 \pm \sqrt{3^2 - 4(2)(-5)}}{2(2)} = \frac{-3 \pm \sqrt{49}}{4}$$

Hence, $x = 1$ and $x = -2{,}5$ are the roots of the parabola. The points $(-2{,}5\ ;\ 0)$ and $(1\ ;\ 0)$ are on the parabola.

If we combine all this information above, we can trace the graph of the parabola $y = 2x^2 + 3x - 5$ precisely.



2. Exponential functions

Definition: we call a function whose form satisfies

$$f\ (x) = a^x \text{, où } a > 0\ \ et\ (a \neq 1)$$

an exponential function of base a.

Particularity: whatever the value of a, an exponential function always passes through the intercept $(0{,}1)$.

Domain: an exponential function is defined for each value of x, hence

$$Dom(a^x) = \mathbb{R}$$

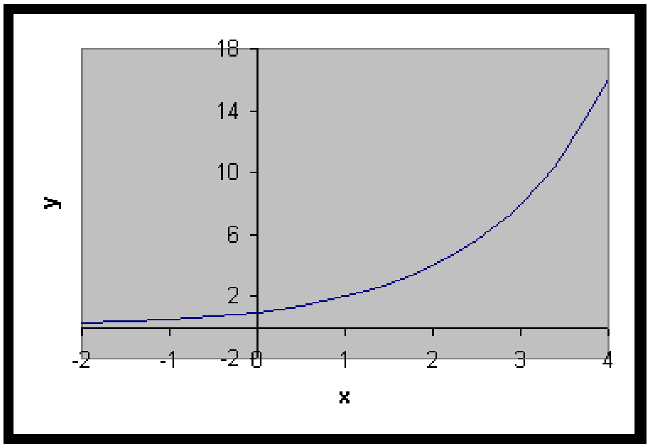
Image: for all values of a and x, an exponential function remains strictly positive, hence

$$Im(a^x) = \{y \mid y > 0\}$$

Behaviour of an exponential function: an exponential function is monotonous. It is

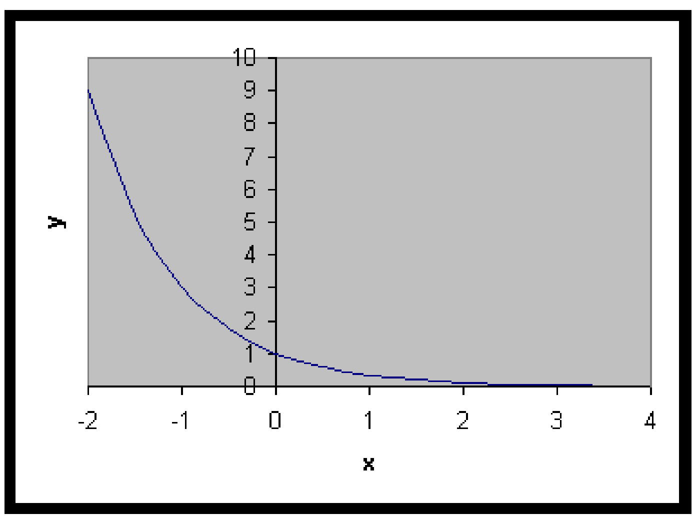
- Strictly increasing if $a > 1$

Graph of the function $y = 2^x$



- Strictly decreasing if $0 < x < 1$

Graph of the function $y = \left(\frac{1}{3}\right)^x$



Remark: an exponential function never has any roots, no matter the value of a.

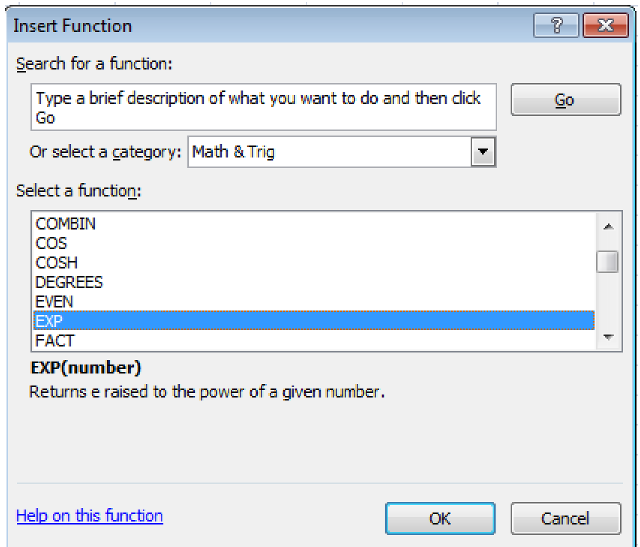
The special value $a = e = 2{,}71828 \ldots$, called the Napierian (or Naperian) or natural base, leads to the best-known and most-used exponential function.

The function $f(x) = e^x$ has the same characteristics as any other exponential function with base $a > 1$.

2.1. Using Excel in calculations with the exponential function e^x

Excel has functions that permit the rapid calculation of exponential functions with Napierian base.

On an Excel worksheet, select the icon $\boldsymbol{f_x}$ and then the category of functions **Math & Trig.** Choose the function EXP



A dialogue box opens and asks you for the value of the exponent.

Exercises

Given the exponential function $f(x) = e^x$. Calculate, using Excel, the following values:

a. $f(1)$
b. $f(3)$
c. $f(-1)$
d. $f(-3/2)$

(answers : $f(1) = e^1 = 2{,}71828$; $f(3) = e^3 = 20{,}0855$; $f(-1) = e^{-1} = 0{,}3679$; $f(\frac{-3}{2}) = e^{\frac{-3}{2}} = 0{,}2231$).

2.2. Law of the exponents

One needs to know a couple properties of exponents to be able to do algebraic manipulations. We list the most important ones:

1) $a^m a^n = a^{m+n}$

2) $a^0 = 1$

3) $(a^m)^n = a^{mn}$

4) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

5) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

6) $\frac{a^m}{a^n} = a^{m-n}$

7) $(ab)^m = a^m b^m$

Exercise

With the help of exponents, simplify the following algebraic expressions:

$a)\ \frac{x^2 x^4}{x^3}$

$b)\ \sqrt{16y^2x^2}$

$c)\ \frac{2x^4y^3}{8y^5x^2}$

$d)\ \frac{z^3(y\sqrt{x})^2}{xyz}$

3. Logarithmic functions

If you knew that $e^x = 3$, how could you find the value of x ? The answer to that question is given in the following definition:

Definition: the natural or Napierian logarithm, $\ln x$, is the inverse function of the exponential function e^x . Hence, $\ln(e^x) = x$ and $e^{lnx} = x$

Domain: The logarithmic function $ln\, x$ is defined for each strictly positive value of x, hence

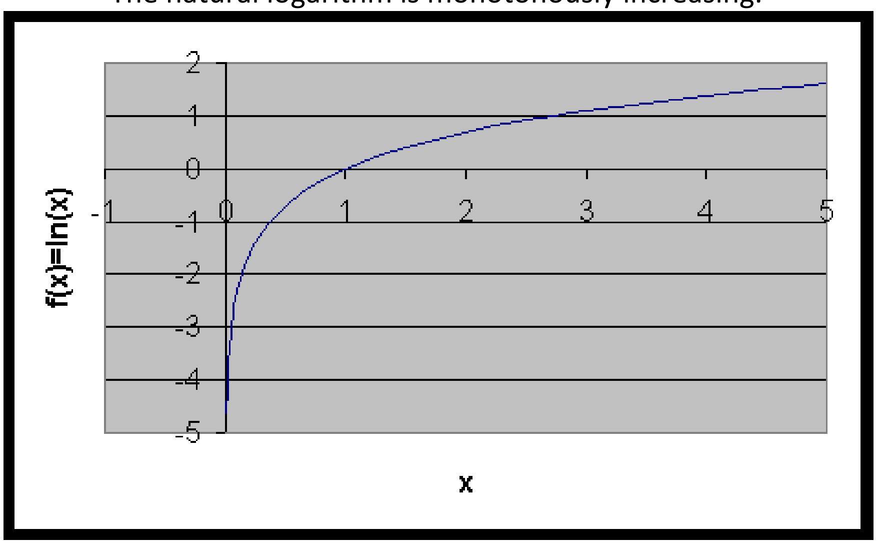
$$Dom(ln\, x) = \{x \mid x > 0\}$$

Image:

$$Im(\ln x) = \mathbb{R}$$

Behaviour of the function $ln\, x$

The natural logarithm is monotonously increasing.



Example

With the help of the definition of the natural logarithm, find the value of x such that the equality $e^x = 3$ is satisfied.

$$e^x = 3 \rightarrow \ln(e^x) = \ln(3)$$

$$x = \ln 3$$

This solution method can be used for more complex equations as well.

Example

Solve the equation $e^{2x+3} \;=\; 10$.

$$e^{2x+3} \;=\; 10 \;\rightarrow \ln(e^{2x+3}) = \ln\,(10)$$

$$\rightarrow 2x + 3 = \ln\,(10)$$

$$\rightarrow 2x = \ln(10) - 3$$

$$\rightarrow x = \frac{ln\,(10)-3}{2}$$

Example

What is the value of x such that $ln(8x - 9) = 20$?

Let us use the fact that the inverse of the function $\ln x$ is e^{x} **:**

$$ln(8x - 9) = 20 \rightarrow \; \mathrm{e}^{\ln\left(e^{8x-9}\right)} = \mathrm{e}^{20}$$

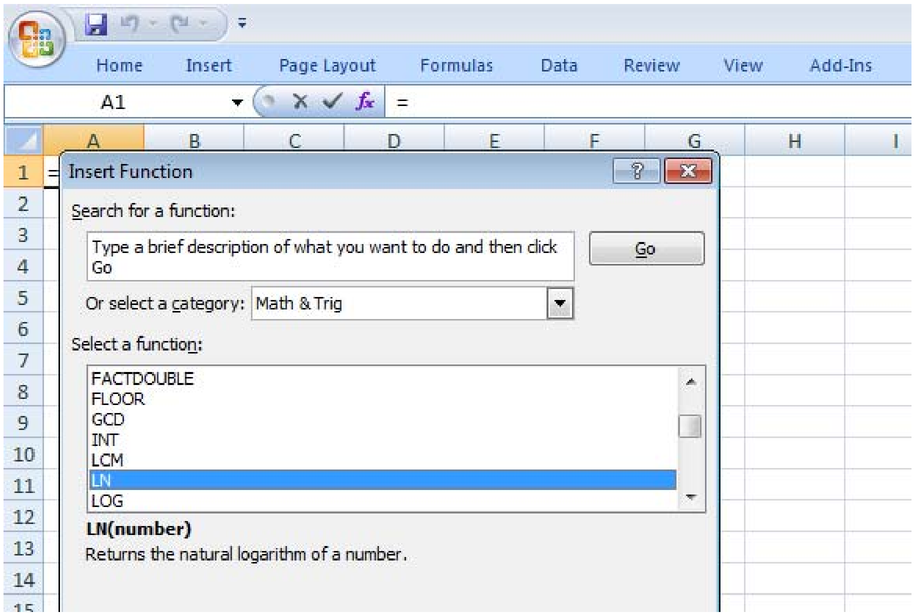
$$\rightarrow 8x - 9 = \mathrm{e}^{20}$$

$$\rightarrow 8x = \mathrm{e}^{20} + 9$$

$$\rightarrow x = \frac{\mathrm{e}^{20} + 9}{8}$$

3.1. Using Excel in calculations with logarithmic functions

Excel has an integrated function $ln(\ )$, that allows for fast calculations of the logarithmic function with Napierian base. On an Excel worksheet, select the icon $\boldsymbol{f_x}$ and the category of functions **Math & Trig.**



A dialogue box will open and ask you what value you want to give to the exponent.

Exercise
Given the logarithmic function $f\ (x)\ =\ ln\ x$. Calculate, using Excel, the following values

a. $f\ (1)$
b. $f\ (3)$
c. $f\ (e^2)$
d. $f\ (-3)$

3.2. Properties of logarithms

Some properties of logarithms are indispensable to simplify certain algebraic expressions or to solve certain equations:

1) $\ln(a.b) = \ln a + \ln b$

2) $ln\left(\frac{a}{b}\right) = \ln a - \ln b$

3) $\ln(a^b) = b.\ln a$

4) $\ln 1\ = 0$

Example

Use the properties of logarithms to find the value of x such that

$$4^x = 24$$

The exponential equation that we have to solve has a base other than e. However, property three allows us to solve it using a trick similar to one used above:

$$4^x = 24 \rightarrow \ln 4^x = \ln(24)$$

$$x . \ln 4 = \ln 24$$

$$x = \frac{\ln 24}{\ln 4} \approx 2.2925$$