SINGLE VARIABLE OPTIMIZATION

Summary

1. Local optimum of a function

1.1. Local maximum

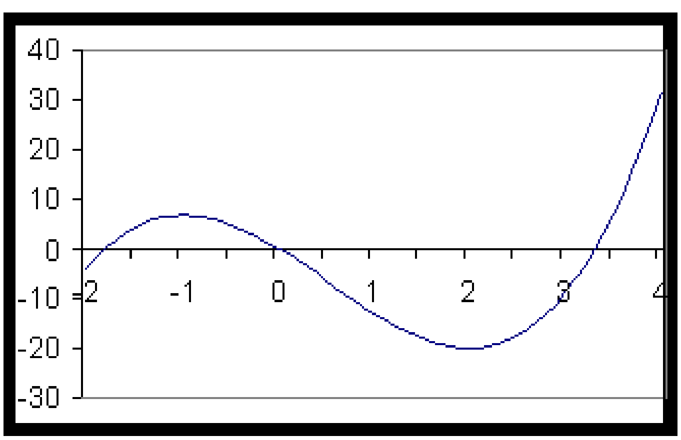
Consider a function $f(x)$. The point x^* is called local maximum if and only if there is I, an open interval containing x^*, such that $f\ (x^*) \geq f\ (x)$ for all x in I.

1.2. Local minimum

Given a function $f(x)$. The point x^* is called a local minimum if and only if there is I, an open interval containing x^*, such that $f\ (x^*) \leq f\ (x)$ for all x in I.

Example

The following graph presents a function that has a local maximum as well as a local minimum.



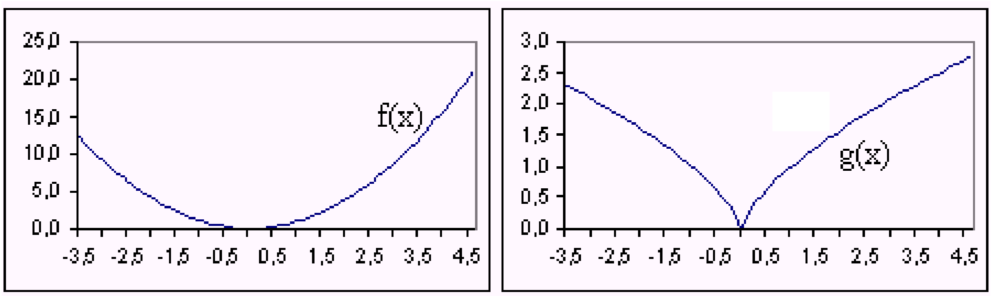
A local maximum is found at the point $x = -1$ since there is an open interval $I1$ containing $x = -1$ on which $f(-1) \geq f(x)$ for all x in $I1$.

A local minimum is found at the point $x = 2$ since there is an open interval $I2$ containing $x = 2$ on which $f(2) \leq f(x)$ for all x in $I2$.

1.3. Stationary points and critical points

In economics, production strategies of a company are determined according to objectives sought out by the company. Sometimes we want to minimize the costs, but usually, we want to maximize profits. Whatever the situation, we are particularly interested in optimal values (maximum or minimum)…

Let us consider the graph of the following functions, identified $f(x)$ and $g(x)$. Each has a local minimum. What can we see at these points ?



We must first notice that both functions cease to decrease and begin to increase at the minimum point (*x* = 0). However, this transition is not made in the same manner for both. The function $f(x)$ goes from decreasing to increasing progressively and at the minimum point, the slope is zero. For $g(x)$, the passage from decreasing to increasing is abrupt, such that the slope is not defined at the minimum. These two types of optima are identified below:

Definition : Stationary point

Consider a continuous function $f(x)$, differentiable at $x = x^*$. The point $x = x^*$ is called a stationary point if the derivative of f is zero at that point, i.e. if $f'(x^*) = 0$.

Example

$f(x) = x2 + 2x - 1 \rightarrow f'(x) = 2x + 2$

The derivative of f is zero when

$2x + 2 = 0 \rightarrow 2x = -2 \rightarrow x = -1$

$x = -1$ is therefore a stationary point of the function f.

Definition : Critical point

Given a function $f(x)$, well defined at $x = x^*$. The point $x = x^*$ is called critical point if the derivative of f does not exist at that point.

Example

$$f(x) = x^{2/3} \rightarrow f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$$

The derivative of f does not exist when $x = 0$ since the denominator then takes the value 0. $x = 0$ is therefore a critical point of f.

1.4. Search for a local optimum

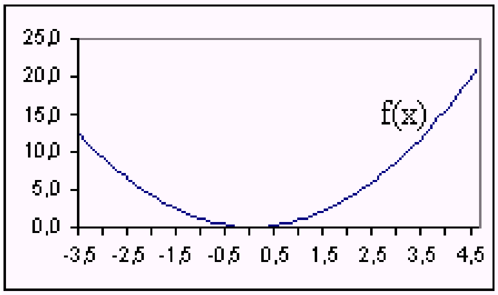
Theorem

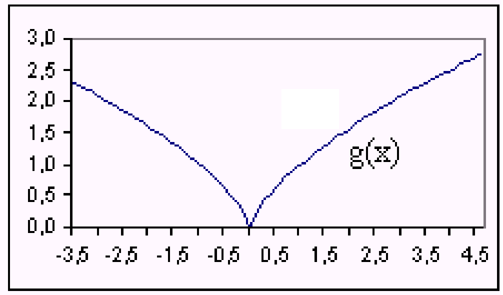
Given a function of x, f defined on an open interval I. A local optimum of f is either a stationary point or a critical point.

What the previous theorem claims is that it is useless to search anywhere else but at the stationary and critical points when looking for a local optimum. Pay attention though! The theorem does not state that all stationary or critical point are local optima... Once the stationary and critical points are found, you must determine the nature of these points to know if they are minima, maxima or neither.

Study of a stationary or critical point using the first derivative

Let us revisit the graphical example that we presented above. The functions $f(x)$ et $g(x)$ both have a local minimum at $x = 0$.





The point $x = 0$ is a stationary point of function $f(x)$. However, $x = 0$ is a critical point of function $g(x)$. Nonetheless, a common phenomenon occurs in both cases: the derivative changes sign at the minimum point. The study of the first derivative allows us to determine the nature of a stationary or critical point.

The first derivative rule

Given the function $\boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{x} = \boldsymbol{x}^*$ a stationary or critical point of the function.

$\boldsymbol{f}\ (\boldsymbol{x}^*)$ is :

- a local minimum if the derivative goes from negative to positive at $\boldsymbol{x}^*$.
- a local maximum if the derivative goes from positive to negative at $\boldsymbol{x}^*$.

Methodology : identification of all the local optima of a function $\boldsymbol{f}(\boldsymbol{x})$

1. Carry out the first derivative of $f(x)$;
2. Find all stationary and critical points ;
3. Draw a table of variations to study the derivative around stationary and critical points ;
4. Conclude.

Example 1

Find all local optima of the function$(x) \;=\; 2x^3 \;-\; 3x^2 \;-\; 12x +\; 4$.

1. **Calculate the first derivative of $f(x)$:**

   $f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2)$

2. **Find all stationary and critical points**

   We obtain a stationary point when $f\ '(x) \;= 0$.

   $6(x^2 - x - 2) = 0$

   $x^2 - x - 2 = 0$

   $(x + 1)(x - 2) = 0 \rightarrow x = -1, x = 2$

   There are therefore two stationary points ($x = -1,\ x = 2$). There are however, no critical points since the derivative is well defined for all x.

3. **Draw a table of variations to study the derivative around stationary and critical points :**

   A table of variations must contain

- all stationary and critical
- the value of the function at stationary and critical points
- the intervals between and around the stationary and critical points
- the sign of the derivative in these intervals

	$\boldsymbol{x < -1}$	$\boldsymbol{x = -1}$	$\boldsymbol{-1 < x < 2}$	$\boldsymbol{x = 2}$	$\boldsymbol{x > 2}$
$\boldsymbol{f(x)}$		11		-16	
$\boldsymbol{f\ '(x)}$	+	0	-	0	+
	Increasing	**Local max**	Decreasing	**Local min**	Increasing

4. Conclude

At point $x \;=\; -1$, we see that the derivative goes from positive to negative. In accordance with the first derivative rule, $f\ (-1) \;=\; 11$ is a local maximum. At point $x \;=\; 2$, we observe that the derivative goes from negative to positive. According to the first derivative rule, $f\ (2) \;=\; -16$ is a local minimum.

Example 2

Find all local optima of the function $f(x) = x^{\frac{1}{3}}(x+1)$

1. **Calculate the first derivative of $f(x)$:**

$$f'(x) = \left(x^{\frac{1}{3}}\right)' . (x+1) + x^{\frac{1}{3}} . (x+1)' \quad \text{(product rule)}$$

$$= \frac{1}{3}x^{-\frac{2}{3}}.(x+1) + x^{\frac{1}{3}}.1$$

$$= \frac{x+1}{3x^{\frac{2}{3}}} + x^{\frac{1}{3}}$$

$$= \frac{x+1}{3x^{\frac{2}{3}}} + \frac{x^{\frac{1}{3}}.3x^{\frac{2}{3}}}{3x^{\frac{2}{3}}} \quad \text{(common denominator)}$$

$$= \frac{x+1+3x}{3x^{\frac{2}{3}}}$$

$$= \frac{1+4x}{3x^{\frac{2}{3}}}$$

2. **Find all stationary and critical points :**

We obtain a stationary point when $f\,'(x) = 0$.

This is obtained when the numerator is zero: $1 + 4x = 0$.

Therefore, $x = -1/4 = -0{,}25$ is a stationary point.

A critical point is obtained when $f\,'(x)$ is not defined. Since the denominator is zero when $\boldsymbol{x} = 0$, this is a critical point.

3. **Draw a table of variations to study the derivative around stationary and critical points :**

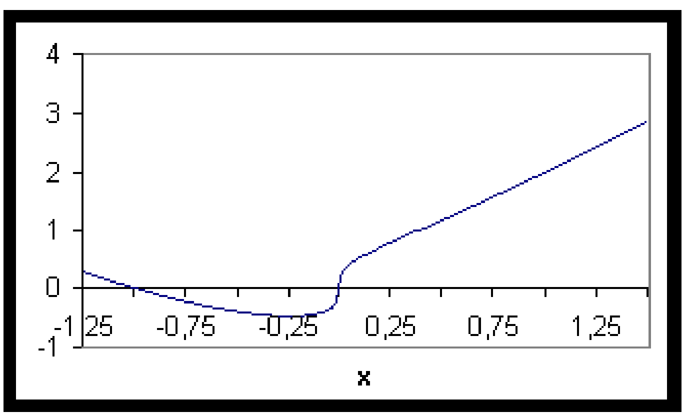
	$x < -1/4$	$x = -1/4$	$-1/4 < x < 0$	$X = 0$	$x > 0$
$f(x)$		-0,4725		0	
$f'(x)$	-	0	+	non définie	+
	Decreasing	**Local min**	Increasing	**Neither min nor max**	Increa-sing

Conclusion:

At the point $x = -1/4$, we observe that the derivative goes from negative to positive. According to the first derivative rule, $f(-1/4) = -0{,}4725$ is a local minimum.

At point $x = 0$, the derivative does not change signs. This critical point is thus neither a local minimum nor a local maximum.

Here is the graph of $(x) = x^{\frac{1}{3}}.(x+1)$:



Observe the abrupt change in direction at $\boldsymbol{x} = 0$ (critical point) where the function does not however cease to grow (we have neither a minimum nor a maximum).

Exercise

For the following functions, find all stationary and critical points, draw a table of variations and determine where the local minima and maxima are found. Using Excel, trace the graph of each function to confirm your results.

a. e^{x^3-3x}
b. $\ln(x^2+1)$
c. $\frac{x}{x^2+1}$