THE STEPS OF THE SIMPLEX ALGORITHM

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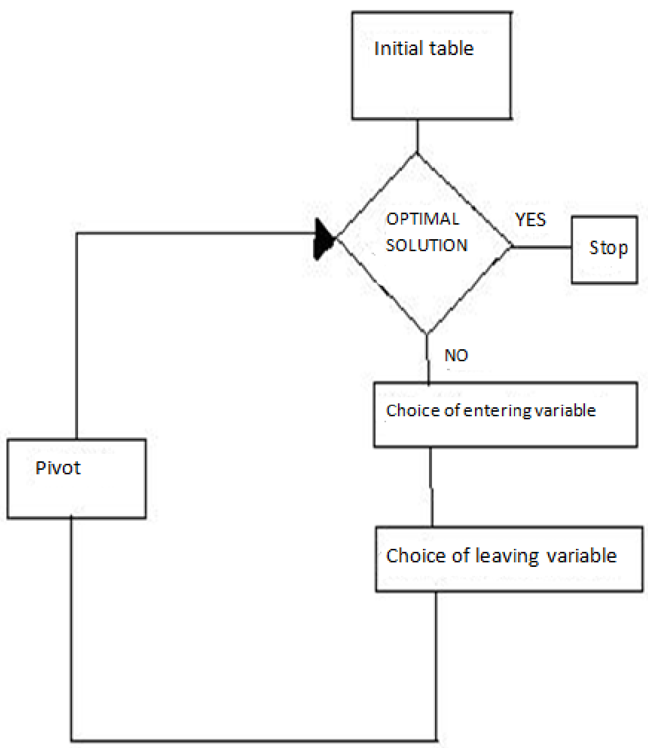

1. Introduction

A linear program (LP) that appears in a particular form where all constraints are equations and all variables are nonnegative is said to be ***in standard form***.

2. Slack and surplus variables

Before the simplex algorithm can be used to solve a linear program, the problem must be written in standard form.

a. Constraints of type ($\leq$) : for each constraint i of this type, we add a ***slack*** variable e_i, such that e_i is nonnegative.

   Example: $3x_1 + 2x_2 \leq 2$ translates into $3x_1 + 2x_2 + e_1 = 2, e_1 \geq 0$

b. Constraints of type ($\geq$) : for each constraint i of this type, we add a ***surplus*** variable e_i, such that e_i is nonnegative.

   Example: $3x_1 + 2x_2 \geq 2$ translates into $3x_1 + 2x_2 - e_2 = 2, e_2 \geq 0$

A linear program that contains (technological) constraints of the type $\leq$ is abbreviated as (LP). A linear program that contains mixed (technological) constraints ($\leq, \geq, =$) is abbreviated as (GP). A linear program (LP) resp. (GP) converted into standard form is abbreviated as (PL=) resp. (PG=).

3. Basic and non-basic variables

Consider a system of equations with n variables and m equations where $n \geq m$. A basic solution for this system is obtained in the following way:

a) Set $n - m$ variables equal to zero. These variables are called non-basic variables (N.B.V).
b) Solve the system for the m remaining variables. These variables are called basic variables (B.V.)
c) The vector of variables obtained is called the basic solution (it contains both basic and non-basic variables).

A basic solution is ***admissible*** if all variables of the basic solution are nonnegative.

It is crucial to have the same number of variables as equations.

4. Admissible solutions

Each basic solution of (LP=) for which all variables are nonnegative, is called an admissible basic solution. This admissible basic solution corresponds to an extreme point (corner solution).

5. Solution of a linear program (LP)

(LP)

Ex : $Max\ Z = 1000\, x_1 + 1200\, x_2$

$$\begin{aligned} s.t.\ 10\, x_1 + 5x_2 &\leq 200 \\ 2x_1 + 3x_2 &\leq 60 \\ x_1 &\leq 34 \\ x_2 &\leq 14 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(LP)

Ex : $Max\ Z = 1000\, x_1 + 1200\, x_2$

$$\begin{aligned} s.t.\ 10\, x_1 + 5x_2 + e_1 &= 200 \\ 2x_1 + 3x_2 + e_2 &= 60 \\ x_1 + e_3 &= 34 \\ x_2 + e_4 &= 14 \\ x_1, x_2, e_1, e_2, e_3, e_4 &\geq 0 \end{aligned}$$

$(n - m) = 0$

$n = 6$ and $m = 4$

$(6 - 4) = 2$ variables $= 0$

Non-basic variables

Basic variables:

if $x_1 = x_2 = 0$ then

$$\begin{aligned} e_1 &= 200 \\ e_2 &= 60 \\ e_3 &= 34 \\ e_4 &= 14 \end{aligned}$$

Step A: initial table

Coef. in Z		1000	1200	0	0	0	0	
Base		X_1	X_2	E_1	E_2	E_3	E_4	b_i
Coef. Z	Basic Var.							
0	E_1	10	5	1	0	0	0	200
0	E_2	2	3	0	1	0	0	60
0	E_3	1	0	0	0	1	0	34
0	E_4	0	1	0	0	0	1	14
z_j		0	0	0	0	0	0	0
$C_j - z_j$		1000	1200	0	0	0	0	

The initial table is written in the following way:

The bleu frame corresponds to the constraints of (LP=).

The green frame corresponds to z_j : the coefficients in $\times\, a_i$.

Example for the column of X_1 called (a_1) :

$$0 \times 10 + 0 \times 2 + 0 \times 1 + 0 \times 0 = 0$$

The pink frames correspond to the coefficients (C_j) of the variables in the objective function (Z).

The grey frame corresponds to the value of the basic variables.

The orange frame corresponds to the value of Z, i.e. the value of the objective function, calculated as follows :

$$0 \times 200 + 0 \times 60 + 0 \times 34 + 0 \times 14 = 0$$

Step B : selection of the entering variable (to the set of basic variables)

Maximum of the $C_j - z_j$ for maximum problems.

Minimum of the $C_j - z_j$ for the minimum problems.

In our example: x_2 has the greatest $C_j - z_j$; hence it enters in the set of basic variables.

Step C : selection of the leaving variable

In a problem of either min **OR** max, the leaving variable is the minimum of

$$\frac{b_i}{a_{ik}}\bigg| a_{ik} > 0$$

In our example, we need to evaluate:

Entering variable

Coef. in Z		1000	1200	0	0	0	0	
Base		X_1	X_2	E_1	E_2	E_3	E_4	b_i
Coef. Z	Basic Var.							
0	E_1	10	5	1	0	0	0	200
0	E_2	2	3	0	1	0	0	60
0	E_3	1	0	0	0	1	0	34
0	E_4	0	1	0	0	0	1	14
z_j		0	0	0	0	0	0	0
$C_j - z_j$		1000	1200	0	0	0	0	

200/5 = 40

60/3 = 20

14/1 = 14 → is the minimum, hence e_4 is the variable that leaves the set of basic variables.

Step D : pivot

Coef. in Z		1000	1200	0	0	0	0	
Base		X_1	X_2	E_1	E_2	E_3	E_4	b_i
Coef. Z	Basic var.							
0	E_1	10	5	1	0	0	0	200
0	E_2	2	3	0	1	0	0	60
0	E_3	1	0	0	0	1	0	34
0	E_4	0	1	0	0	0	1	14
z_j		0	0	0	0	0	0	0
$C_j - z_j$		1000	1200	0	0	0	0	

The blue cell is called the pivot. To go to the next table (and hence to carry out the first iteration), it is essential to use the pivot.

Pivoting goes like this:

One starts by dividing the line of the pivot by the pivot.

In our example, we divide by 1.

Coef. in Z		1000	1200	0	0	0	0	
Base		X_1	X_2	E_1	E_2	E_3	E_4	b_i
Coef. Z	Basic var.							
0	E_1							
0	E_2							
0	E_3							
1200	X_2	0	1	0	0	0	1	14
z_j		0	0	0	0	0	0	0
$C_j - z_j$		1000	1200	0	0	0	0	

We continue to construct the identity matrix for the basic variables. We write one the intersection of these variables and zero elsewhere.

Coef. in Z		1000	1200	0	0	0	0	
Base		X_1	X_2	E_1	E_2	E_3	E_4	b_i
Coef. Z	Basic var.							
0	E_1		0	1	0	0		
0	E_2		0	0	1	0		
0	E_3		0	0	0	1		
1200	X_2	0	1	0	0	0	1	14
z_j		0	0	0	0	0	0	0
$C_j - z_j$		1000	1200	0	0	0	0	

We need to calculate the values for the remaining cells from the previous table (or the initial table for the first iteration).

Coef. in Z		1000	1200	0	0	0	0	
Base		X_1	X_2	E_1	E_2	E_3	E_4	b_i
Coef. Z	Basic var.							
0	E_1		0	1	0	0		
0	E_2		0	0	1	0		
0	E_3		0	0	0	1		
1200	X_2	0	1	0	0	0	1	14
	z_j	0	0	0	0	0	0	0
	$C_j - z_j$	1000	1200	0	0	0	0	

Initial table:

Coef. in Z		1000	1200	0	0	0	0	
Base		X_1	X_2	E_1	E_2	E_3	E_4	b_i
Coef. Z	Basic var.							
0	E_1	10	5	1	0	0	0	200
0	E_2	2	3	0	1	0	0	60
0	E_3	1	0	0	0	1	0	34
0	E_4	0	1	0	0	0	1	14
	z_j	0	0	0	0	0	0	0
	$C_j - z_j$	1000	1200	0	0	0	0	

In our example, the 10 in the red-framed cell is calculated with the following formula

$$10 - \frac{element\ on\ the\ line\ of\ the\ pivot * element\ in\ the\ column\ of\ the\ pivot}{pivot}$$

Hence, $10 - \frac{0*5}{1} = 10$.

Let us calculate the green-framed cell. We obtain -3 in the following way:

$$0 - \frac{3 * 1}{1} = -3$$

Coef. in Z		1000	1200	0	0	0	0	
Base		X_1	X_2	E_1	E_2	E_3	E_4	b_i
Coef. Z	Basic var.							
0	E_1	10	0	1	0	0	-5	
0	E_2	2	0	0	1	0	-3	
0	E_3	1	0	0	0	1	0	
1200	X_2	0	1	0	0	0	1	14
z_j		0	0	0	0	0	0	0
$C_j - z_j$		1000	1200	0	0	0	0	

The remaining cells are calculated in the same way. When the table is full (such as the one below), one can continue to the second iteration, that will be carried out in the same way.

6. Stopping criterion

We stop when we reach the optimality criterion. The simplex algorithm stops when:

- $C_j - z_j \leq 0$ for a maximum problem
- $C_j - z_j \geq 0$ for a minimum problem